E.g. $\frac{d}{dx}\int_{x^2}^{x^3} e^{t^2}\,dt = ?$

$$\int_{x^2}^{x^3} e^{t^2}\,dt = \int_a^{x^3} e^{t^2} - \int_a^{x^2} e^{t^2}\,dt$$

Fundamental thm of Calculus

$$\frac{d}{dy}\int_a^y e^{t^2}\,dt = e^{y^2}$$

let $u = x^3$, $v = x^2$. $\frac{du}{dx} = 3x^2$. $\frac{dv}{dx} = 2x$

$$\frac{d}{dx}\int_a^{x^3} e^{t^2}\,dt = \frac{d}{dx}\int_a^u e^{t^2}\,dt$$

$$= \frac{du}{dx}\,\frac{d}{du}\int_a^u e^{t^2}\,dt$$

$$= 3x^2\,e^{u^2} = 3x^2\,e^{x^6}$$

$$\frac{d}{dx}\int_a^{x^2} e^{\tau^2}\,dt = \frac{d}{dx}\int_a^v e^{t^2}\,d\tau$$

$$= \frac{dv}{dx}\,\frac{d}{dv}\int_a^v e^{t^2}\,dt$$

$$= 2x\,e^{v^2} = 2x\cdot e^{x^4}$$

$$\frac{d}{dx}\int_{x^2}^{x^3} e^{t^2}\,dt = \frac{d}{dx}\left(\int_a^{x^3} e^{t^2}\,dt - \int_a^{x^2} e^{\tau^2}\,dt\right)$$

$$= \frac{d}{dx}\int_a^{x^3} e^{t^2}\,dt - \frac{d}{dx}\int_a^{x^2} e^{\tau^2}\,d\tau$$

$$= 3x^2\,e^{x^6} - 2x\,e^{x^4} \quad \square$$

E.g, $\displaystyle\int_{-2}^{1} \frac{2x+1}{x^2+x+1}\,dx$

proper rational function

x^2+x+1 irreducible

let $u = x^2+x+1 \leftarrow$ $u(-2) = 4-2+1 = 3$
$u(1) = 1+1+1 = 3$

$\frac{du}{dx} = 2x+1$

$$\int_{-2}^{1} \frac{2x+1}{x^2+x+1}\,dx = \int_{u(-2)}^{u(1)} \frac{du}{u} = \int_3^3 \frac{du}{u} = 0$$

E.g, $\displaystyle\int_2^3 \frac{dx}{(x-1)(x^2+2x-3)}$

$\frac{1}{(x-1)(x^2+2x-3)}$ is a proper rational function

$x^2+2x-3 = (x-1)(x+3)$

$$\frac{1}{(x-1)^2(x+3)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+3}$$

$$= \frac{a(x-1)(x+3)}{(x-1)(x-1)(x+3)} + \frac{b(x+3)}{(x-1)^2(x+3)} + \frac{c(x-1)^2}{(x+3)(x-1)^2}$$

$$= \frac{a(x-1)(x+3)+b(x+3)+c(x-1)^2}{(x-1)^2(x+3)}$$

$1 = a(x-1)(x+3)+b(x+3)+c(x-1)^2$

plug in $x=1$ $\quad 1 = 0 + 4b + 0 \rightsquigarrow b = \frac{1}{4}$

plug in $x=-3$ $\quad 1 = 0 + 0 + c\cdot 16 \rightsquigarrow c = \frac{1}{16}$

$u(3) = 2 \quad v(3) = 6$

$u(2) = 1 \quad v(2) = 5$

" $x = 0$ $\quad 1 = a \cdot (-3) + 3b + C$

$= -3a + \frac{3}{4} + \frac{1}{16}.$

$u = x-1$

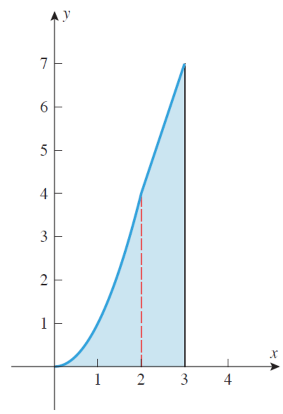
$v = x+3$

$a = \cdots$

$$\int_2^3 \frac{1}{(x-1)^2(x+3)}\,dx = a\int_2^3 \frac{1}{x-1}\,dx + \frac{1}{4}\int_2^3 \frac{1}{(x-1)^2}\,dx + \frac{1}{16}\int_2^3 \frac{1}{x+3}\,dx$$

$$= a \ln u \Big|_{u(2)}^{u(3)} + \frac{1}{4}\left(-\frac{1}{u}\right)\Big|_1^2 + \frac{1}{16}\ln v\Big|_5^6$$

$= \cdots$



$$\begin{aligned}\int_0^3 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx \qquad \text{(integrate separately)}\\ &= \frac{x^3}{3}\Big|_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6}.\end{aligned}$$

Exercise 2.1.

1. $\int_0^1 2xe^{x^2}\,dx = e - 1.$
2. $\int_{-1}^2 |x|\,dx = \frac{5}{2}.$

Example 2.6. Compute $\frac{d}{dx}$ for (1) $\int_1^x e^{t^2}\,dt$, (2) $\int_{x^2}^{x^3} e^{t^2}\,dt$, (3) $\int_{g(x)}^{h(x)} f(t)\,dt$.

Solution. It's impossible to get explicit formula for $F(t) = \int e^{t^2}\,dt$.

1. By fundamental theorem of calculus (1), we have

$$\frac{d}{dx}\int_1^x e^{t^2}\,dt = e^{x^2}.$$

2. Let $F'(t) = e^{t^2}$, then

$$\frac{d}{dx}\int_{x^2}^{x^3} e^{t^2}\,dt = \frac{d}{dx}(F(x^3) - F(x^2)) = F'(x^3)\cdot 3x^2 - F'(x^2)\cdot 2x = e^{x^6}\cdot 3x^2 - e^{x^4}\cdot 2x.$$

$$\int_{x^2}^{x^3} e^{t^2}dt = \int_a^{x^3} e^{t^2}dt - \int_a^{x^2} e^{t^2}dt$$

let $u = x^3$
$v = x^2$.

$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt - \frac{d}{dx}\int_a^{g(x)} f(t)\,dt$

$= \frac{dh}{dx}\frac{d}{dh}\int_a^{h} f(t)\,dt - \frac{dg}{dx}\frac{d}{dg}\int_a^{g} f(t)\,d\tau$

$= h' f(h(x)) - g' f(g(x))$ □

3. Let $F'(t) = f(t)$,

$$\begin{aligned}
\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt &= \frac{d}{dx}(F(h(x)) - F(g(x))) \\
&= F'(h(x))\cdot h'(x) - F'(g(x))\cdot g'(x) \\
&= f(h(x))h'(x) - f(g(x))g'(x).
\end{aligned}$$

■

Exercise 2.2. $\frac{d}{dx}\int_{2x}^{x+1} e^{\sqrt{t}}\,dt = e^{\sqrt{x+1}} - 2e^{\sqrt{2x}}$.

3 Definite Integration by Substitution & Integration by Parts

Theorem 3.1.

$$\boxed{\int_a^b f(g(x))g'(x)\,dx \;\overset{g(x)=u}{=\!=}\; \int_{g(a)}^{g(b)} f(u)\,du}$$

let $u = x^2+1$

$\frac{du}{dx} = 2x$

Example 3.1.

$\int_{x=0}^{x=1} 8x\,(x^2+1)\,dx = \int_{u(0)}^{u(1)} 4u\,du = 2u^2 + C$

$= 2(x^2+1)^2 + C$

1.

$$\begin{aligned}
\int_0^1 8x(x^2+1)dx &= \int_0^1 4(x^2+1)\,d(x^2+1) \\
&= \int_1^2 4u\,du \quad (x^2+1=u,\ (0)^2+1=1,\ 1^2+1=2) \\
&= 2u^2\big|_1^2 \\
&= 2\times 2^2 - 2\times 1^2 = 6.
\end{aligned}$$

FoC: $\int_0^1 8x\,(x^2+1)\,dx$

$= 2(x^2+1)^2\Big|_0^1$

$= 2\cdot 2^2 - 2\cdot 1^2$

$= 6$ □

2.

$$\begin{aligned}
\int_e^{e^2} \frac{1}{x\ln x}dx &= \int_e^{e^2}\frac{1}{\ln x}\,d(\ln x) \\
&= \int_1^2 \frac{1}{u}du \quad (\ln x = u,\ \ln e = 1,\ \ln e^2 = 2) \\
&= \ln u\big|_1^2 \\
&= \ln 2 - \ln 1 = \ln 2.
\end{aligned}$$

$\int_1^2 \frac{1}{u}\,du$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$u(e) = \ln e = 1$

$u(e^2) = \ln e^2 = 2$

$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$

Theorem 3.2.

$$\boxed{\int_a^b u(x)\,d(v(x)) = u(x)v(x)\Big|_a^b - \int_a^b v(x)\,d(u(x))}$$

Example 3.2.

$u = \ln x \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x \quad v = \frac{x^2}{2}$

1.

$$\begin{aligned}\int_1^e x\ln x\,dx &= \int_1^e \ln x\,d\left(\frac{x^2}{2}\right)\\ = uv\Big|_1^e - \int v\frac{du}{dx}\,dx &= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\,d\ln x\\ &= \left(\frac{e^2}{2}\ln e - \frac{1}{2}\ln 1\right) - \int_1^e \frac{x}{2}\,dx\\ &= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e\\ &= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)\\ &= \frac{e^2}{4} + \frac{1}{4}.\end{aligned}$$

$\frac{1}{x}\,dx$

2.

$u \frac{dv}{dx} = uv\Big|_0^1 - \int_0^1 v\frac{du}{dx}\,dx$

$u = x \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^x \quad v = e^x$

$$\begin{aligned}\int_0^1 xe^x\,dx &= \int_0^1 x\,d(e^x)\\ &= xe^x|_0^1 - \int_0^1 1\cdot e^x\,dx\\ &= e - e^x|_0^1 = 1\end{aligned}$$

$e^1 - e^0 = e - 1$

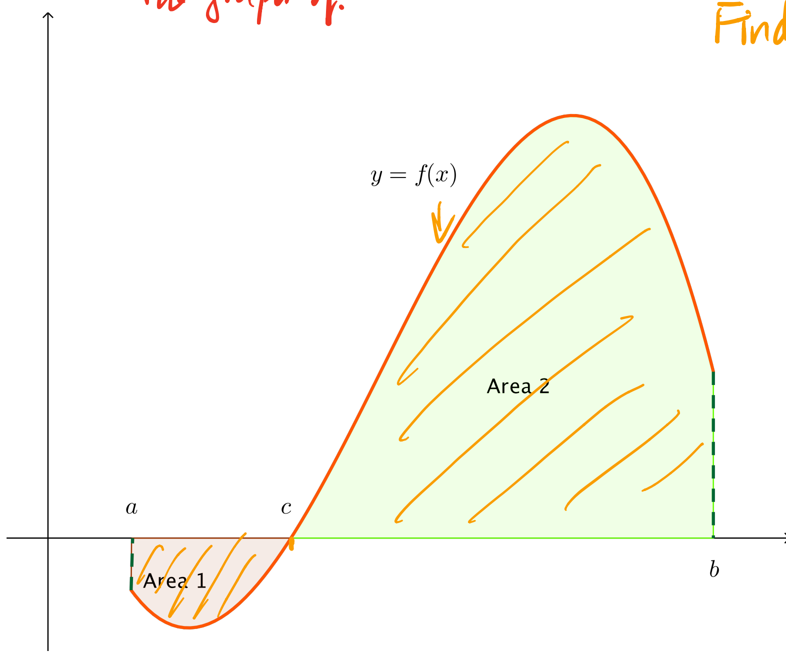
Exercise 3.1. 1. $\int_{-2}^{1} \frac{2x+1}{x^2+x+1}dx = ?$

2. $\int_2^3 \frac{dx}{(x-1)(x^2+2x-3)} = ?$

4 Application of Definite Integration

$\int_a^b f(x)\,dx$ = signed area.
= area2 − area1

4.1 Area bounded by the graph of $f(x)$ and x-axis on $[a, b] = \int_a^b |f(x)|\,dx$



Find the total area (not the signed area) of the shaded region
= Area 2 + Area 1
$= \int_c^b f(x)dx - \int_a^c f(x)dx$
$= \int_a^b |f(x)|dx$

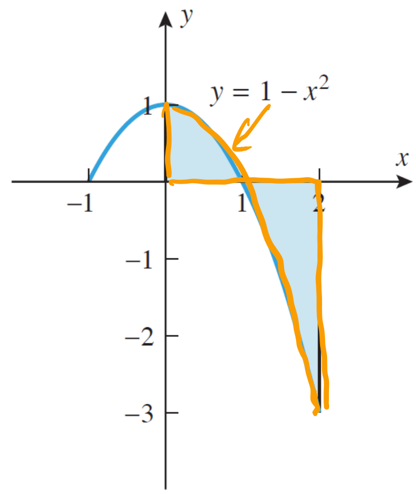
$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = -\text{ Area 1} + \text{ Area 2 } = \text{Signed area}}$$

$$\boxed{\int_a^b |f(x)|\,dx = \int_a^c -f(x)\,dx + \int_c^b f(x)\,dx = \text{ Area 1} + \text{ Area 2 } = \text{Area}}$$

Example 4.1. Find the total area between the curve $y = \overset{f(x)}{1 - x^2}$ and the x-axis over the interval $[0, 2]$.

Solution. Let $1 - x^2 = 0, \quad \Rightarrow \quad x = \pm 1.$

$$1 - x^2 \begin{cases} \geq 0, & \text{for } -1 \leq x \leq 1, \\ < 0, & \text{for } x < -1 \quad \text{or} \quad x > 1. \end{cases}$$



The area is given by

$$\begin{aligned}\int_0^2 |1-x^2|\,dx &= \int_0^1 (1-x^2)dx + \int_1^2 -(1-x^2)dx \\ &= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2 \\ &= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2.\end{aligned}$$

■

Exercise 4.1. Area bounded by the graph of $f(x) = x - \sqrt{x}$ and x-axis on $[0, 2]$.

$$\int_0^2 |x-\sqrt{x}|\,dx = \int_0^1 (\sqrt{x}-x)\,dx + \int_1^2 (x-\sqrt{x})\,dx$$

$x - \sqrt{x} \geq 0$ when $x \geq 1$

≤ 0 when $0 \leq x \leq 1$

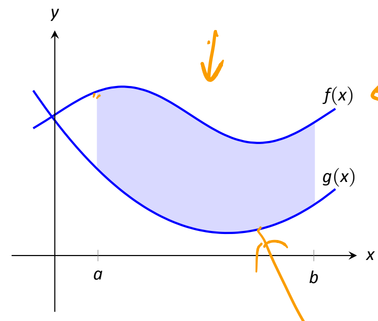
4.2 Area bounded by the graphs of $f(x)$, $g(x)$ on $[a, b]$ = $\int_a^b |f(x) - g(x)|\,dx$

Theorem 4.1. *Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is*

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Proof. The area between $f(x)$ and $g(x)$ is obtained by subtracting the area under g from the area under f. Thus the area is

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx.$$

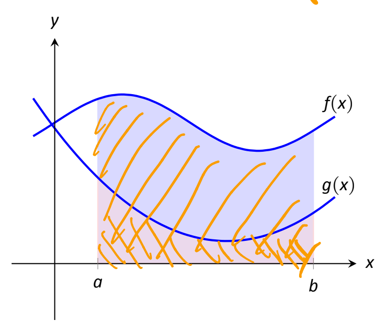


← graph. $y = f(x)$

← $y = g(x)$

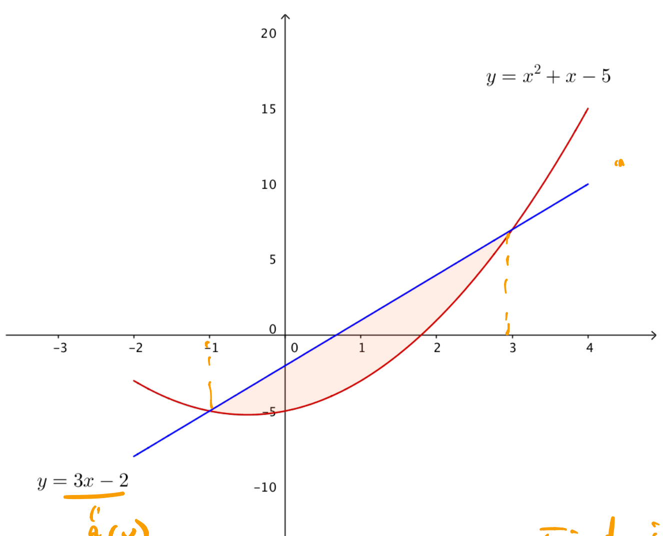
area of this region

$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$



□

Example 4.2. Find the area of the region enclosed by $\underbrace{y = x^2 + x - 5}$ $\overset{f(x)}{}$ and $\underbrace{y = 3x - 2}$ $\overset{g(x)}{}$.



$g(x)$

Find intersection pts of the two graphs.

at an intersection pt.

$(x, y) = (x, f(x))$

$= (x, g(x))$

so $f(x) = g(x)$ at intersection pts

Solution. Let $\underbrace{x^2 + x - 5}_{f(x)} = \underbrace{3x - 2}_{g(x)} \quad \Rightarrow \quad x = -1, 3.$

$x^2 - 2x - 3 = 0$

The area is

$$\int_{-1}^{3}\big((3x-2)-(x^2+x-5)\big)\,dx = \int_{-1}^{3}(-x^2+2x+3)\,dx$$
$$= \left(-\frac{1}{3}x^3+x^2+3x\right)\Bigg|_{-1}^{3}$$
$$= -\frac{1}{3}(27)+9+9-\left(\frac{1}{3}+1-3\right)$$
$$= 10\frac{2}{3}.$$

■

Example 4.3. Find the area bounded by

$$y=f(x)=x, \quad y=g(x)=\frac{2}{x+1}, \quad \text{and} \quad y=h(x)=2x+2.$$

$y = h(x) = 2x+2$

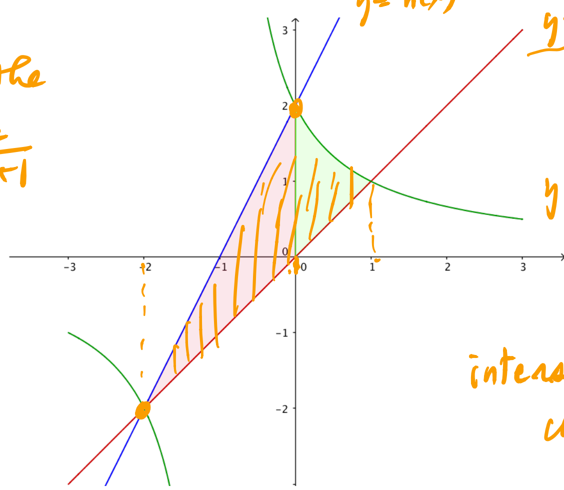
$y = x = f(x)$

red curve: $y = x = f(x)$ intersects the green curve $y = g(x) = \frac{2}{x+1}$ where $x = \frac{2}{x+1}$.

$\rightsquigarrow\ x(x+1) = 2.$

$\rightsquigarrow\ x^2+x-2=0$

$\rightsquigarrow\ x = -2, 1$



$y = g(x) = \frac{2}{x+1}$

intersection pts of the green curve $y = g(x) = \frac{2}{x+1}$ and the blue curve $y = h(x) = 2x+2$

here $\frac{2}{x+1} = 2x+2$

i.e. $2 = 2(x+1)^2$

$1 = (x+1)^2$

$\rightsquigarrow\ x+1 = \pm 1$

$x = 0, -2$

orange region green region

Solution. Area is

$$\int_{-2}^{0}(h(x)-f(x))dx+\int_{0}^{1}(g(x)-f(x))dx$$
$$=\int_{-2}^{0}(2x+2-x)+\int_{0}^{1}\left(\frac{2}{x+1}-x\right)dx$$
$$=\left[\frac{x^2}{2}+2x\right]_{-2}^{0}+\left[2\ln|x+1|-\frac{x^2}{2}\right]_{0}^{1}$$
$$=2+(2\ln 2-\frac{1}{2})=\frac{3}{2}+\ln 4.$$

■

4.3 Other Applications

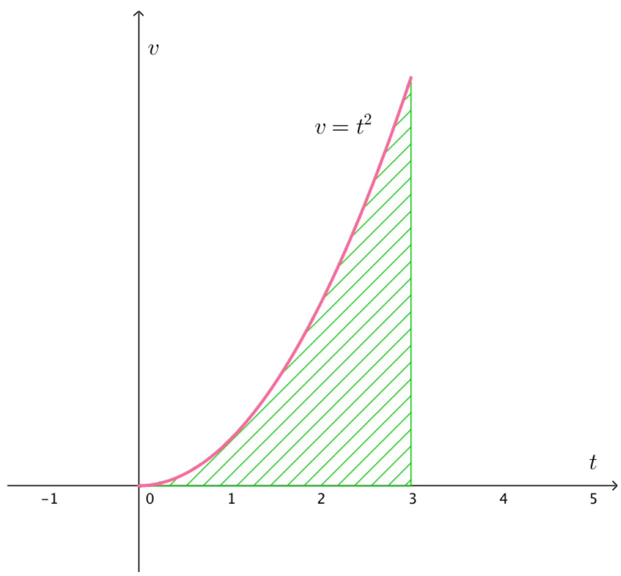
Example 4.4. An object moves along x-axis towards right with speed $v(t) = t^2$ m/s. Calculate the distance traveled from $t = 0$ to $t = 3$s.

Solution. Let $S(t)$ be the position at t. Then, $S'(t) = v(t) = t^2$.

The distance from $t = 0$ to $t = 3$ is

$$\underbrace{S(3) - S(0)}_{\text{total distance change}} = \int_0^3 \overbrace{S'(t)}^{\text{rate of change}} \, dt = \int_0^3 t^2 \, dt = \frac{1}{3}t^3 \Big|_0^3 = 9\text{m}$$

Geometrically,



∎

Example 4.5. Let $L(t)$ be the level of carbon monoxide (CO). Given that $L'(t) = 0.1t + 0.1$ parts per million (ppm). How much will the pollution change from $t = 0$ to $t = 3$?

Solution.
$$L(3) - L(0) = \int_0^3 L'(t)dt = \left[0.05t^2 + 0.1t\right]_0^1 = 0.75\text{ppm}.$$

∎

Exercise 4.2. Let t be the time (in hour). Let $m(t)$ be the mass of a certain amount of protein. The protein is changed to an amino acid and cause a decrease in mass at a rate

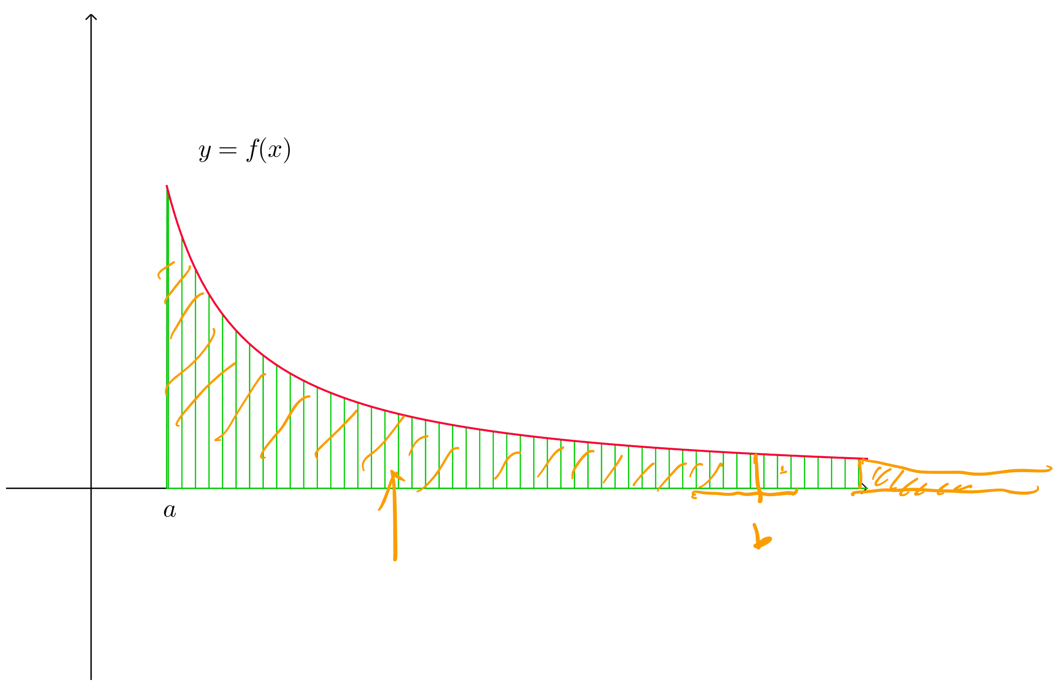
$$\frac{dm}{dt} = \frac{-2}{t+1}\text{g/hr}.$$

Find the decrease in mass of the protein from $t = 2$ to $t = 5$.

Ans: $-2\ln 2$.

5 Improper Integrals

Question: How to find area of an unbounded region?



Definition 5.1. The following types of integrals are called "improper integrals". The integrals we have encountered previously, namely integrals of piecewise continuous functions over finite intervals, are "proper integrals".

Define

1.
$$\boxed{\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx}$$

if the limit exists, we say that the integral is **convergent**. Otherwise, **divergent**.

2.
$$\boxed{\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx}$$

if the limit exists, we say that the integral is **convergent**. Otherwise, **divergent**.

3. Let c be a fixed real number.

$$\boxed{\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx}$$

if both the two integrals on the right are convergent, we say that the integral is **convergent**. Otherwise, **divergent**.

Example 5.1.

1. $\int_0^{+\infty} e^{-x}dx = \lim_{b\to+\infty}\int_0^b e^{-x}dx = \lim_{b\to+\infty} -e^{-x}\big|_0^b = \lim_{b\to+\infty}(e^0 - e^{-b}) = \lim_{b\to+\infty}(1-e^{-b}) =$ 1, convergent.

border line case

2. $\int_1^{+\infty}\frac{1}{x}dx = \lim_{b\to+\infty}\int_1^b \frac{1}{x}dx = \lim_{b\to+\infty}\ln x\big|_1^b = \lim_{b\to+\infty}(\ln b - \ln 1) = \lim_{b\to+\infty}\ln b = +\infty$, divergent.

3. $\int_1^{+\infty}\frac{1}{x^2}dx = \lim_{b\to+\infty}\int_1^b\frac{1}{x^2}dx = \lim_{b\to+\infty}\left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to+\infty}\left(1-\frac{1}{b}\right) = 1$, convergent. □

4. $\int_1^{+\infty}\frac{1}{\sqrt{x}}dx = \lim_{b\to+\infty}\int_1^b\frac{1}{\sqrt{x}}dx = \lim_{b\to+\infty}2\sqrt{x}\Big|_1^b = \lim_{b\to+\infty}2(\sqrt{b}-1) = +\infty$, divergent.

5. $\int_{-\infty}^{0} e^x dx$ $= \lim_{a\to-\infty}\int_a^0 e^x dx = \lim_{a\to-\infty} e^x\Big|_a^0 = \lim_{a\to-\infty}(e^0 - e^a)$

$= 1$ □

$a>0$ $\quad\int_a^{\infty}\frac{1}{x^n}dx$ is convergent when $n>1$

divergent when $x\leq 1$

Example 5.2. Compute $\displaystyle\int_0^{+\infty} \frac{dx}{(x+1)(3x+2)}$.

Solution.

$$\frac{1}{(x+1)(3x+2)} = \frac{3}{3x+2} - \frac{1}{x+1}.$$

Hence

$$\int_0^b \frac{dx}{(x+1)(3x+2)} = [\ln|3x+2| - \ln|x+1|]_0^b$$

$$= \ln|3b+2| - \ln|b+1| - \ln|2| = \ln\frac{|3b+2|}{|b+1|} - \ln 2.$$

Because

$$\lim_{b\to+\infty} \frac{|3b+2|}{|b+1|} = \lim_{b\to+\infty} \frac{|3b+2| \times \frac{1}{|b|}}{|b+1| \times \frac{1}{|b|}}.$$

$$\lim_{b\to+\infty} \frac{\left|3+\frac{2}{b}\right|}{\left|1+\frac{1}{b}\right|} = \frac{3}{1} = 3.$$

Therefore

$$\lim_{b\to+\infty} \int_0^b \frac{dx}{(x+1)(3x+2)} = \ln 3 - \ln 2.$$

■

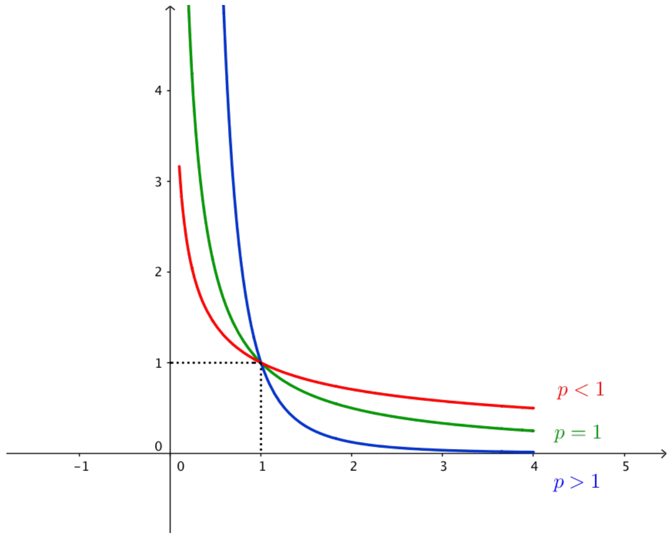
Exercise 5.1. Let $p > 1$. Prove that

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1}, & \text{if} \quad p > 1, \qquad \text{convergent} \\ +\infty, & \text{if} \quad 0 < p \le 1, \quad \text{divergent}. \end{cases}$$

Remark. From the above exercise,

1. $\displaystyle\lim_{x\to+\infty} f(x) = 0 \quad \not\Rightarrow \quad \int_1^{+\infty} f(x)\,dx$ is convergent.
2. For all $p > 0$, $\dfrac{1}{x^p} \to 0$ as $x \to +\infty$. However, only for $p > 1$, $\dfrac{1}{x^p}$ decays fast enough to guarantee the total area $\displaystyle\int_1^{+\infty} \frac{1}{x^p}\,dx$ is finite.

Remark. All the integration techniques can be applied, e.g. integration by substitution,...



Example 5.3. Compute $\int_{-\infty}^{1} xe^x\,dx$. (integration by parts)

Solution.

$$\int_{-\infty}^{1} xe^x\,dx = \lim_{a\to-\infty}\int_a^1 xe^x\,dx.$$

$$\int xe^x\,dx = \int x d(e^x) = xe^x - \int e^x\,dx = (x-1)e^x + C.$$

$$\begin{aligned}\int_{-\infty}^{1} xe^x\,dx &= \lim_{a\to-\infty}(x-1)e^x|_a^1\\ &= \lim_{a\to-\infty}(1-a)e^a \quad \infty\cdot 0 \quad \text{indeterminate form}\\ &= \lim_{a\to-\infty}\frac{1-a}{e^{-a}} \quad \frac{\infty}{\infty}\\ &= \lim_{a\to-\infty}\frac{-1}{-e^{-a}} \quad \text{L'Hôpital's rule}\\ &= 0.\end{aligned}$$

■

Exercise 5.2. $\int_{-\infty}^{1} x^2e^x dx = e$